# Quantum gates

## Quantum computing

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Visualizing qubit states

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Measurement lab

## Last time

We introduced the basic unit of quantum information: the **qubit**.

Quick review:

https://www.wooclap.com/QCOMP1

# Visualizing qubit states

Consider $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \in \mathcal{V}_2$

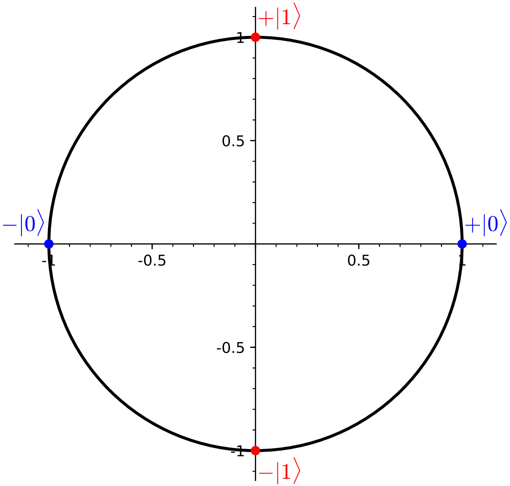
In general $\alpha$ and $\beta$ are complex numbers: $\dim_{\mathbb{R}} \mathcal{V}_2 = 4$

Hard to visualize! Let us assume for the moment that $\alpha, \beta \in \mathbb{R}$.

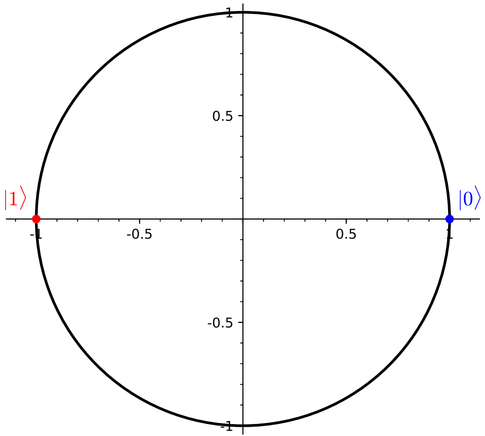
Since $|\psi\rangle \sim \frac{1}{\|\psi\|}|\psi\rangle$, we can assume without loss of generality that $\alpha^2 + \beta^2 = 1$.

Looks like a circle...

# A circle ?



# Yes: the Bloch circle



## The (real) Bloch representation

According to the first picture we are tempted to write:

$$|\psi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle \qquad 0 \leq \theta < 2\pi$$

but there is the ambiguity $\theta \longleftrightarrow \theta + \pi$, $|\psi\rangle \sim -|\psi\rangle$.

In the second, more accurate picture, what we actually see is the point

$$P_{|\psi\rangle} = (\cos 2\theta,\, \sin 2\theta).$$

Thus it would have been better to write, non-ambiguously,

$$|\psi\rangle = \cos(\tfrac{\theta}{2})\,|0\rangle + \sin(\tfrac{\theta}{2})\,|1\rangle.$$

## Angle between two states

In the (real) Bloch representation:

$$\begin{cases} |\psi\rangle = \cos(\frac{\theta}{2})\,|0\rangle + \sin(\frac{\theta}{2})\,|1\rangle, \\ |\varphi\rangle = \cos(\frac{\phi}{2})\,|0\rangle + \sin(\frac{\phi}{2})\,|1\rangle \end{cases}$$

we have

$$\langle\varphi\,|\,\psi\rangle = \cos(\tfrac{\phi}{2})\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\phi}{2})\sin(\tfrac{\theta}{2}) = \cos\tfrac{\phi-\theta}{2}.$$

In particular:

$$\langle\varphi\,|\,\psi\rangle = 0 \iff \tfrac{\phi-\theta}{2} = \pm\tfrac{\pi}{2} \iff \phi = \theta \pm \pi.$$

*Orthogonal* states lie *opposite* on the Bloch circle.

## Towards the Bloch representation

Now for a general state $0 \neq |\psi\rangle \in \mathcal{V}_2$:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle, \qquad \alpha, \beta \in \mathbb{C}.$$

Without loss of generality we can assume $|\alpha|^2 + |\beta|^2 = 1$ (normalized state).

Equivalent normalized states: if $|\psi\rangle \sim |\varphi\rangle$, then $|\psi\rangle = \gamma\,|\varphi\rangle$ with $|\gamma| = 1$.

So: if $\alpha = A\,e^{ia}$, by multiplying by $e^{-ia}$ we can reduce to the case

$$\alpha = A \text{ is real},\ \beta = B\,e^{ib},\ A^2 + B^2 = 1.$$

## Bloch representation

$$|\psi\rangle = A\,|0\rangle + B\,e^{ib}\,|1\rangle, \quad A^2 + B^2 = 1.$$

From the real case we know we should write $A = \cos(\frac{\theta}{2})$, $B = \sin(\frac{\theta}{2})$.

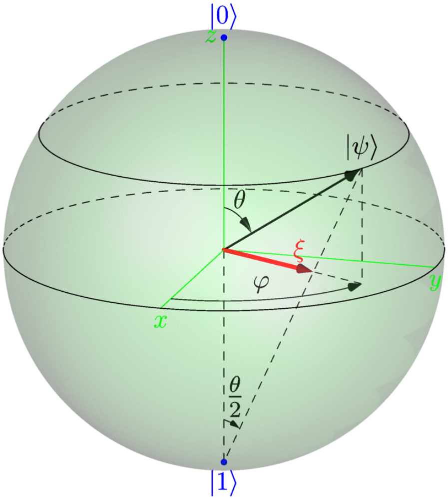
We have proved:

*Every qubit state is equivalent to a unique normalized state of the form*

$$\cos(\tfrac{\theta}{2})\,|0\rangle + \sin(\tfrac{\theta}{2})\,e^{i\phi}\,|1\rangle.$$

These correspond to points $(\cos\phi\,\sin\theta, \sin\phi\,\sin\theta, \cos\theta)$ on a *sphere*.

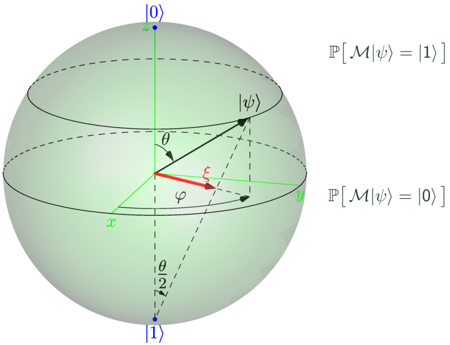
# The Bloch sphere $\mathcal{B}$ (click title for interactive model)




http://stla.github.io/stlapblog/posts/BlochSphere.html

# Properties of the Bloch representation

- Pairs of orthogonal states correspond to antipodal points on the Bloch sphere.
- The probability that $|\psi\rangle$ is measured as $|0\rangle$ or $|1\rangle$ can be interpreted as relative areas on the sphere.



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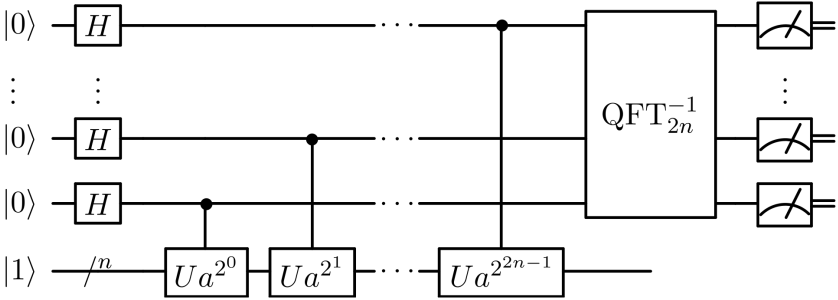
# Spoilers ahead: Shor's algorithm



# Quantum circuits

Quantum circuits are made up of

- **quantum registers** containing qubits
- **quantum logic gates** modifiying the state of these qubits
- **classical registers** containing regular bits
- **measurements** mapping quantum registers to classical registers

that can then be manipulated with a classical electronic circuit.

# NOT gate

$$\begin{cases} \text{NOT}\,|0\rangle = |1\rangle \\ \text{NOT}\,|1\rangle = |0\rangle \end{cases}$$

$$\text{NOT}(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|1\rangle + \beta\,|0\rangle$$

$$\text{NOT}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \qquad \text{NOT} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

## Interpretation on the Bloch sphere

Fixed points of NOT:

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

NOT can be thought of as a *rotation of* $\pi$ *around the x-axis*

often called the **Pauli X** gate for this reason and written NOT, $X$ or $\oplus$

Note: $X^2 = I$

## Exercise

Geometrical interpretation of the quantum gates

$$Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

## Hadamard gate

$$H = \frac{X + Z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Sends $|0\rangle$ to $H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|1\rangle$ to the orthogonal state $H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$.

Remark: $H^2 = I$ (isn't it?)

## **Phase gate** $P = P(\theta)$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

$$P|0\rangle = |0\rangle, \qquad P|1\rangle = e^{i\theta}$$

$$P(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|0\rangle + e^{i\theta}\,\beta\,|1\rangle$$

Remark : $P(\pi) = Z$

## Universal gate $U$

Depends on 3 parameters $\theta$, $\phi$ and $\lambda$:

$$U = \begin{bmatrix} \cos(\frac{\theta}{2}) & -e^{i\lambda}\sin(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) & e^{i(\lambda+\phi)}\cos(\frac{\theta}{2}) \end{bmatrix}$$

*All* gates encountered so far are special cases !

Remark: $U$ is a unitary matrix ($U^\dagger U = I$)

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# Lab 1: Single qubit measurements

Consider on IBM Q the $U$ gate with parameters $(\theta, \phi, \lambda)$.

1) If $U|0\rangle$ is measured, what are the probabilities of observing $|0\rangle$ and $|1\rangle$?
2) Using the parameters provided on this spreadsheet, verify that the Statevector, Measurement probabilities and representation on the Bloch sphere correspond to what you would expect.
3) Add a measurement 

to your circuit and verify that the *simulated* results agree with what you would expect ("run" the circuit on `ibmq_qasm_simulator`).
4) Then, submit your circuit to a *real* quantum computer (don't hold your breath) and record the experimental probability $p$ of observing 0 on 1024 measurements.
5) Submit a short report of your results on Teams by next class!