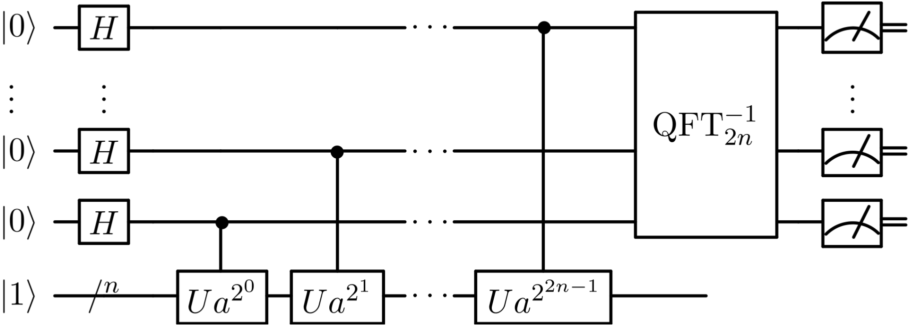
Quantum states

Quantum computing

G. Chênevert

M1 · S2 (2020)

By the end of this class...



Quantum states

Review of quantum formalism

Qubits

Bloch sphere

Measurement lab

Recall: Quantum states

The states of a quantum system can be written as linear combinations

$$\phi(\mathbf{x}, t) = \sum_n A_n \, e^{-i \frac{E_n}{\hbar} t} \, \varphi_n(\mathbf{x})$$

where the φ_n are eigenfunctions for the reduced Hamiltonian operator:

$$\widehat{H} \, \varphi_n = E_n \, \varphi_n.$$

These eigenstates are orthogonal with respect to the Hermitian product

$$\langle \varphi \,|\, \psi \rangle = \int \varphi(\mathbf{x})^* \, \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

Braket notation

The instantaneous states form a (complex) vector space $\mathcal{V}$ spanned by the φ_n.

Hermitian product: if the φ_n are **normalized** ($\|\varphi_n\| = \sqrt{\langle \varphi_n | \varphi_n \rangle} = 1$) then for

$$\varphi = \sum_n \alpha_n \varphi_n, \qquad \psi = \sum_n \beta_n \varphi_n,$$

we have

$$\langle \varphi \,|\, \psi \rangle = \sum_n \alpha_n^* \beta_n = \begin{bmatrix} \alpha_0 & \alpha_1 & \dots & \end{bmatrix}^* \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \end{bmatrix} = |\varphi\rangle^\dagger \, |\psi\rangle$$

NB: In QM, $*$ = complex conjugation, $\dagger$ = conjugate + transpose

Measurement

When we measure a **mixed state**

$$|\varphi\rangle = \sum_n \alpha_n |\varphi_n\rangle \in \mathcal{V} \setminus \{\mathbf{0}\} :$$

it gets projected on the **pure state** $|\varphi_n\rangle$ with probability

$$\mathbb{P}\big[\,\mathcal{M}|\varphi\rangle = |\varphi_n\rangle\,\big] = \frac{|\langle\varphi\,|\,\varphi_n\rangle|^2}{\|\varphi\|^2} = \frac{|\alpha_n|^2}{\|\varphi\|^2}.$$

If $|\varphi\rangle$ is normalized, this is just

$$\mathbb{P}\big[\,\mathcal{M}|\varphi\rangle = |\varphi_n\rangle\,\big] = |\langle\varphi\,|\,\varphi_n\rangle|^2 = |\alpha_n|^2.$$

(Sometimes it will be useful to work with not necessarily normalized states.)

Exercise

We measure the mixed quantum state

$$|\varphi\rangle = |\varphi_0\rangle + (3+4i)|\varphi_1\rangle + 7|\varphi_2\rangle + 5i|\varphi_3\rangle.$$

What to we expect to see ?

Answer:

$$\mathbb{P}\left[\,\mathcal{M}|\varphi\rangle = |\varphi_n\rangle\,\right] = \begin{cases} 1\% & n = 0 \\ 25\% & n = 1 \\ 49\% & n = 2 \\ 25\% & n = 3 \end{cases}$$

Equivalence

When two states are proportional: $|\varphi\rangle = \alpha|\psi\rangle$ ($\alpha \neq 0$) then

$$\mathbb{P}\big[\,\mathcal{M}|\varphi\rangle = |\varphi_n\rangle\,\big] = \frac{|\langle\varphi\,|\,\varphi_n\rangle|^2}{\|\varphi\|^2} = \frac{|\alpha|^2\,|\langle\psi\,|\,\varphi_n\rangle|^2}{|\alpha|^2\,\|\psi\|^2} = \mathbb{P}\big[\,\mathcal{M}|\psi\rangle = |\varphi_n\rangle\,\big]$$

Thus $|\varphi\rangle$ and $|\psi\rangle$ cannot be distinguished by measurements: we write $|\varphi\rangle \sim |\psi\rangle$.

Quantum states should really be thought of as *equivalence classes of vectors*

$$\{\,\alpha|\varphi\rangle \;\mid\; \alpha \neq 0\,\}$$

i.e. lines in $\mathcal{V}$: elements of what the mathematicians call the **projective space** $\mathbb{P}^1(\mathcal{V})$.

Simple quantum systems

N-**level** quantum system: when $\dim_{\mathbb{C}} \mathcal{V} = N$.

Notation: we write $\mathcal{V}_N$ for the standard N-level state space with pure states

$$|0\rangle,\ |1\rangle,\ \ldots,\ |N-1\rangle.$$

$N = 1$: $|\varphi\rangle = \alpha\,|0\rangle \sim |0\rangle$

"constant system" that behaves classically

$N = 2$: **qubit** $|\varphi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

Quantum states

Review of quantum formalism

Qubits

Bloch sphere

Measurement lab

Qubit states

In general, the state of a qubit can be thought of as a nonzero linear combination

$$|\varphi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \qquad \alpha, \beta \in \mathbb{C}.$$

When we measure it:

$$\mathbb{P}\big[\,\mathcal{M}|\varphi\rangle = |0\rangle\,\big] = \frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2}, \qquad \mathbb{P}\big[\,\mathcal{M}|\varphi\rangle = |1\rangle\,\big] = \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2}.$$

For a normalized state, $|\alpha|^2 + |\beta|^2 = 1$ so this is just

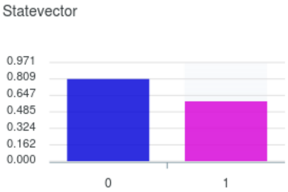
$$\mathbb{P}\big[\,\mathcal{M}|\varphi\rangle = |0\rangle\,\big] = |\alpha|^2, \qquad \mathbb{P}\big[\,\mathcal{M}|\varphi\rangle = |1\rangle\,\big] = |\beta|^2.$$

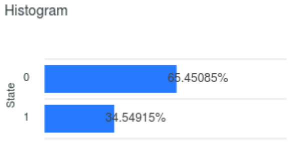
Example

$$|\varphi\rangle = \cos\left(\tfrac{\pi}{5}\right)|0\rangle + \sin\left(\tfrac{\pi}{5}\right) e^{\frac{\pi}{3} i} |1\rangle$$
$$\approx 0.809\,|0\rangle + (0.294 + 0.509 i)\,|1\rangle$$

Normalized state with

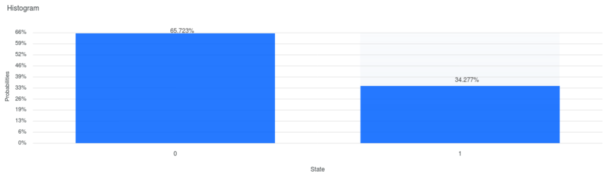
$$\mathbb{P}[\,\mathcal{M}|\varphi\rangle = |0\rangle\,] = \cos^2\left(\tfrac{\pi}{5}\right) \approx 65.5\%, \qquad \mathbb{P}[\,\mathcal{M}|\varphi\rangle = |0\rangle\,] = \sin^2\left(\tfrac{\pi}{5}\right) \approx 34.5\%$$



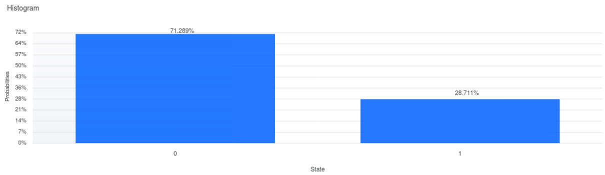


IBM Q Experience results

Result of 1024 *simulations*:



Result of 1024 *executions* on `ibmq_essex`:



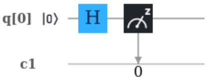
Your turn

Now would be a good time to create an account and start messing around with the

IBM Q Experience

https://quantum-computing.ibm.com/

Suggestion:



yields $$|\varphi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Quantum states

Review of quantum formalism

Qubits

Bloch sphere

Measurement lab

Visualizing qubit states

Consider $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathcal{V}_2$

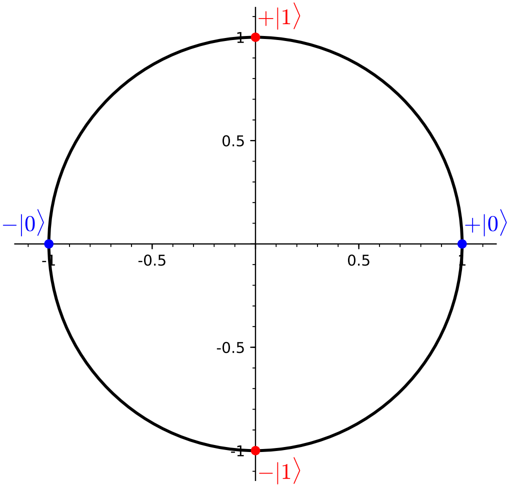
In general α and β are complex numbers: $\dim_{\mathbb{R}} \mathcal{V}_2 = 4$

Hard to visualize! Let us assume for the moment that $\alpha, \beta \in \mathbb{R}$.

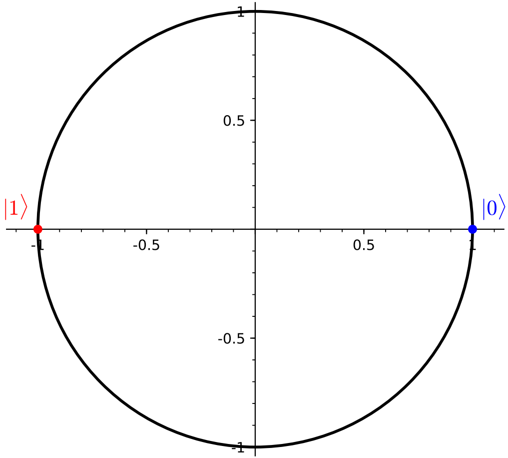
Since $|\varphi\rangle \sim \frac{1}{\|\varphi\|}|\varphi\rangle$, we can assume without loss of generality that $\alpha^2 + \beta^2 = 1$.

Looks like a circle...

A circle ?



Yes: the Bloch circle



The (real) Bloch representation

According to the first picture we are tempted to write:

$$|\varphi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle \qquad 0 \leq \theta < 2\pi$$

but there is the ambiguity $\theta \longleftrightarrow \theta + \pi$, $|\varphi\rangle \sim -|\varphi\rangle$.

In the second, more accurate picture, what we actually see is the point

$$P_{|\varphi\rangle} = (\cos 2\theta, \sin 2\theta).$$

Thus it would have been better to write, non-ambiguously,

$$|\varphi\rangle = \cos(\tfrac{\theta}{2})\,|0\rangle + \sin(\tfrac{\theta}{2})\,|1\rangle.$$

Angle between two states

In the (real) Bloch representation:

$$\begin{cases} |\varphi\rangle = \cos(\frac{\theta}{2})\,|0\rangle + \sin(\frac{\theta}{2})\,|1\rangle, \\ |\psi\rangle = \cos(\frac{\phi}{2})\,|0\rangle + \sin(\frac{\phi}{2})\,|1\rangle \end{cases}$$

we have

$$\langle \varphi \,|\, \psi \rangle = \cos(\tfrac{\theta}{2})\cos(\tfrac{\phi}{2}) + \sin(\tfrac{\theta}{2})\sin(\tfrac{\phi}{2}) = \cos\frac{\theta-\phi}{2}.$$

In particular:

$$\langle \varphi \,|\, \psi \rangle = 0 \iff \frac{\theta-\phi}{2} = \pm\frac{\pi}{2} \iff \phi = \theta \pm \pi.$$

Orthogonal states lie *opposite* on the Bloch circle.

Towards the Bloch representation

Now for a general state $0 \neq |\varphi\rangle \in \mathcal{V}_2$:

$$|\varphi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle, \qquad \alpha, \beta \in \mathbb{C}.$$

Without loss of generality we can assume $|\alpha|^2 + |\beta|^2 = 1$ (normalized state).

Equivalent normalized states: if $|\varphi\rangle \sim |\psi\rangle$, then $|\psi\rangle = \gamma\,|\varphi\rangle$ with $|\gamma| = 1$.

So: if $\alpha = A\,e^{ia}$, by multiplying by e^{-ia} we can reduce to the case

$$\alpha = A \text{ is real, } \beta = B\,e^{ib},\ A^2 + B^2 = 1.$$

Bloch representation

$$|\varphi\rangle = A\,|0\rangle + B\,e^{ib}\,|1\rangle, \quad A^2 + B^2 = 1.$$

From the real case we know we should write $A = \cos(\frac{\theta}{2})$, $B = \sin(\frac{\theta}{2})$.

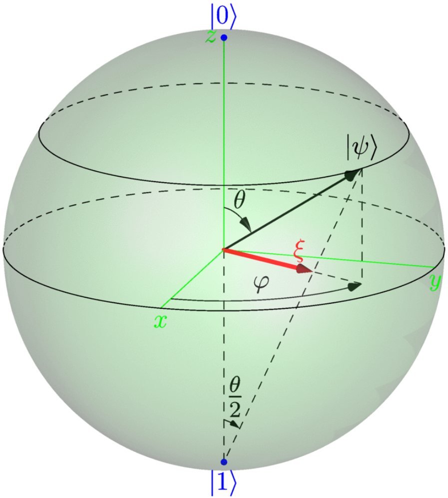
We have proved:

Every qubit state is equivalent to a unique normalized state of the form

$$\cos(\tfrac{\theta}{2})\,|0\rangle + \sin(\tfrac{\theta}{2})\,e^{i\phi}\,|1\rangle.$$

These correspond to points $(\cos\phi\,\sin\theta, \sin\phi\,\sin\theta, \cos\theta)$ on a *sphere*.

The Bloch sphere



http://stla.github.io/stlapblog/posts/BlochSphere.html

Quantum states

Lab 1: Single qubit measurements

In IBM Q, you can use the U_3 gate with parameters $(\theta, \phi, 0)$ to turn $|0\rangle$ into the state with Bloch parameters (θ, ϕ).

1) *Simulate* the measurements of 20 qubits with varying parameters (θ, ϕ) and record the probabilities of getting $|0\rangle$.

2) Plot your simulation results against what the theory predicts (simulated points *vs.* theoretical curve). How do they compare?

3) Measure on a real quantum computer (at least) *one* qubit with random parameters (θ, ϕ) and record the result in this spreadsheet. How does this measurement compare with what you expect?

4) Submit your report on Campus before next class!