Quantum gates

Quantum computing

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Quantum gates

Recall: Bloch sphere

Single qubit gates

Multiple qubit gates

Quantum states

Qubit states are (nonzero) complex linear superpositions of $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \qquad (\alpha, \beta \in \mathbb{C})$$

up to equivalence:

$$|\psi\rangle \sim |\phi\rangle \iff |\psi\rangle = \gamma\,|\phi\rangle \qquad (\gamma \neq 0).$$

Normalized states

Every state can be normalized to an equivalent normalized state:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \qquad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1.$$

Equivalence becomes:

$$|\psi\rangle \sim |\phi\rangle \iff |\psi\rangle = e^{i\theta}\,|\phi\rangle \qquad (\theta \in \mathbb{R}).$$

global phase is irrelevant

Bloch representation

Every qubit state $|\psi\rangle$ is equivalent to a *unique* state of the form

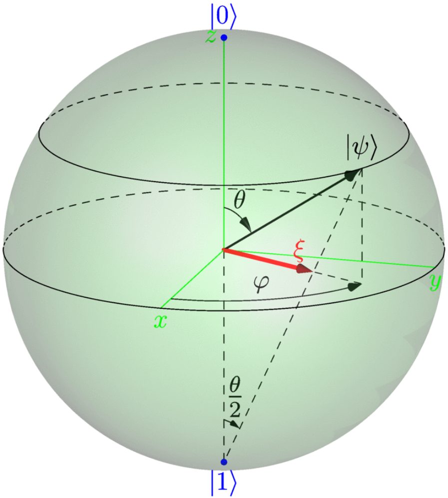
$$\cos(\tfrac{\theta}{2})\,|0\rangle + \sin(\tfrac{\theta}{2})\,e^{i\varphi}\,|1\rangle$$

that correspond bijectively to a point on the unit sphere in $\mathbb{R}^3$

$$P_{|\psi\rangle} = (\cos\varphi\,\sin\theta,\ \sin\varphi\,\sin\theta,\ \cos\theta)$$

$$0 \le \varphi < 2\pi, \quad 0 \le \theta \le \pi$$

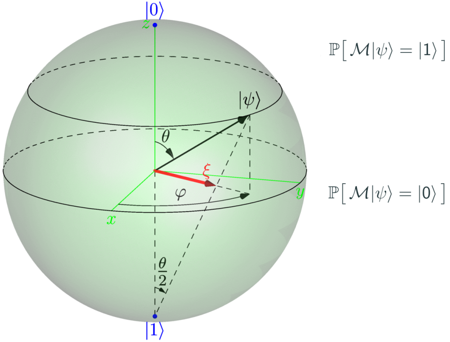
Bloch sphere $\mathcal{B}$



http://stla.github.io/stlapblog/posts/BlochSphere.html

Properties of the Bloch representation

- Pairs of orthogonal states correspond to antipodal points on the Bloch sphere.
- The probability that $|\psi\rangle$ is measured as $|0\rangle$ or $|1\rangle$ can be interpreted as relative areas on the sphere.



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Quantum circuits

Quantum circuits are made up of

- **quantum registers** containing qubits
- **quantum logic gates** modifiying the state of these qubits
- **classical registers** containing regular bits
- **measurements** mapping quantum registers to classical registers

that can then be manipulated with a classical electronic circuit.

NOT gate

$$\begin{cases} \text{NOT} |0\rangle = |1\rangle \\ \text{NOT} |1\rangle = |0\rangle \end{cases}$$

$$\text{NOT}(\alpha |0\rangle + \beta |1\rangle) = \alpha |1\rangle + \beta |0\rangle$$

$$\text{NOT} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \qquad \text{NOT} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Interpretation on the Bloch sphere

Fixed points of NOT:

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

NOT can be thought of as a *rotation of* π *around the x-axis*

often called the **Pauli X** gate for this reason and written NOT, X or $\oplus$

Note: $X^2 = I$

Exercises

1. Write down matrices Y and Z for rotations of π around the y- and z- axis on the Bloch sphere.
2. Verify that $X^2 \sim Y^2 \sim Z^2 \sim XYZ \sim I$
3. Simplify the following circuit:

$$|q_0\rangle - \boxed{X} - \boxed{Z} - \boxed{Y} - \boxed{Y} - \boxed{Z} - \boxed{X} - \boxed{Z} - \boxed{X} -$$

4. Why is this matrix called *the square root of* NOT ?

$$\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

The Hadamard gate

$$H = \frac{X + Z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

5. Verify that H is a unitary matrix: $H^\dagger H = I$.

6. What is the interpretation of H as a rotation on the Bloch sphere?

7. Explain how the H gate can be used to measure a qubit in the $|\pm\rangle$ basis.

Other useful single qubit gates

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{\pi i}{4}} \end{bmatrix}$$

$$\text{and more generally } R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

8. What special cases of R_θ have we encountered yet?

9. Compute HZH.

10. How would you create gates $R_x(\theta)$, $R_y(\theta)$, $R_z(\theta)$ representing rotations around the x-, y- and z-axes?

Summary

- (nonequivalent) qubit states = points on the Bloch sphere
- X, Y, Z, H, S, T, R_θ gates

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} \qquad R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

- Kahoot quiz

General single qubit gate

QM tells us that qubit manipulations preserve orthogonality between states

i.e. can be represented by **unitary** matrices

$$\langle G\psi \,|\, G\phi\rangle = \langle \psi \,|\, \phi\rangle \iff G^\dagger G = I$$

In general we have $|\det G| = 1$; up to matrix equivalence we may assume $\det G = 1$.

Then $G^{-1} = G^\dagger$ for $N = 2$ means

$$G = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Special unitary group

$$\mathrm{SU}_2(\mathbb{C}) = \left\{ \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Two such matrices G_1 and G_2 are equivalent $\iff G_1 = \pm G_2$.

Thus the set (group) of single qubit gates is

$$\mathrm{SU}_2(\mathbb{C})/\{\pm I\}$$

a 3-dimensional geometric space (Lie group)

General single qubit gate

Any single qubit gate G admits an orthogonal eigenbasis $|\psi_0\rangle$, $|\psi_1\rangle$ for which

$$\begin{cases} G\,|\psi_0\rangle = e^{+i\sigma}\,|\psi_0\rangle \\ G\,|\psi_1\rangle = e^{-i\sigma}\,|\psi_1\rangle \end{cases}$$

If $U\,|0\rangle = |\psi_0\rangle$ and $U\,|1\rangle = |\psi_1\rangle$, then

$$U^\dagger G U = \begin{bmatrix} e^{+i\sigma} & 0 \\ 0 & e^{-i\sigma} \end{bmatrix} \sim R_{2\sigma}.$$

On the Bloch sphere, G *is a rotation of angle* 2σ *around the axis through the orthogonal states* $|\psi_0\rangle$ *and* $|\psi_1\rangle$.

Other point of view

Consider the images

$$\begin{cases} |\phi_0\rangle = G\,|0\rangle \\ |\phi_1\rangle = G\,|1\rangle \end{cases}$$

and write Bloch parameters

$$|\phi_0\rangle = \cos(\tfrac{\theta}{2})\,|0\rangle + \sin(\tfrac{\theta}{2})\,e^{i\varphi}\,|1\rangle.$$

Then $|\phi_1\rangle \sim -\sin(\frac{\theta}{2})\,|0\rangle + \cos(\frac{\theta}{2})\,e^{i\varphi}\,|1\rangle$ with phase factor, say, $e^{i\lambda}$

$$G = \begin{bmatrix} |\phi_0\rangle & |\phi_1\rangle \end{bmatrix} = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2})\,e^{i\lambda} \\ \sin(\frac{\theta}{2})\,e^{i\varphi} & \cos(\frac{\theta}{2})\,e^{i(\varphi+\lambda)} \end{bmatrix} = U_3(\theta, \varphi, \lambda)$$

Two points of view

- axis **u** and rotation angle σ
- image of vertical axis **z** and phase parameter λ

The relationship between these two representations is a bit complicated...

Unless one is willing to work with quaternions

$$\mathbb{H} = \{a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \mid a, b, c, d \in \mathbb{R}\}.$$

Universal family

Remark: every single qubit gate G can be expressed as a combination of

$$H \qquad \text{and} \qquad R_\theta \qquad (\theta \in \mathbb{R}) \qquad \text{only.}$$

Idea:

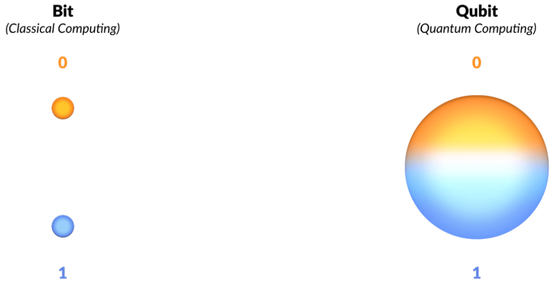
- express G as a combination of $R_x(\alpha)$, $R_y(\beta)$, $R_z(\gamma)$
- explicit formulas for these 3 kinds of rotations

Corollary: every single qubit gate G can be *approximated* by a combination of

$$H \qquad \text{and} \qquad R_{2\pi/n} \qquad (n \gg 0) \qquad \text{only.}$$

Great!

You now understand all possible programs that can run on `imbq_armonk`



$$\mathbb{Z}/2\mathbb{Z} = \{I, X\} \qquad \textit{vs.} \qquad \mathrm{SU}_2(\mathbb{C})/\{\pm I\} = \{U_3(\theta, \phi, \lambda)\}_{\theta,\phi,\lambda} = \mathrm{SO}_3(\mathbb{R})$$

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Two qubit gates

Do we have the analogues of the classical AND, OR, XOR, NAND, ... gates for quantum bits?

No! They lose information...

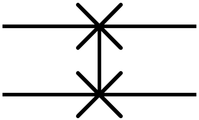
Recall: the space of quantum states for a system of 2 qubits is

$$\mathcal{V}_2 \otimes \mathcal{V}_2 \cong \mathcal{V}_4$$

basis $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, $|1\rangle \otimes |1\rangle$ or $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ or $|0\rangle$, $|1\rangle$, $|2\rangle$, $|3\rangle$

2-qubit gates are represented by 4×4 *unitary matrices*

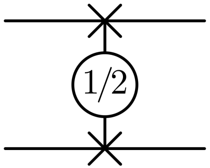
SWAP gate



$$|\psi\rangle \otimes |\phi\rangle \;\mapsto\; |\phi\rangle \otimes |\psi\rangle$$

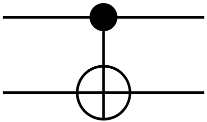
$$\mathrm{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{diag}(1, X, 1)$$

Square root of SWAP



$$\sqrt{\text{SWAP}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{diag}(1, \sqrt{X}, 1)$$

CNOT = CX gate



$$\mathrm{CX}(|x\rangle \otimes |y\rangle) \text{ " = " } \begin{cases} |y\rangle & \text{if } |x\rangle = |0\rangle \\ X|y\rangle & \text{if } |x\rangle = |1\rangle \end{cases} = |x \oplus y\rangle$$

To be able to go back we must output $|x\rangle$ as well:

$$\text{"}\mathrm{CX} \begin{bmatrix} |x\rangle \\ |y\rangle \end{bmatrix} = \begin{bmatrix} |x\rangle \\ |x\rangle \oplus |y\rangle \end{bmatrix}\text{"}$$

CNOT = CX gate

$$\mathrm{CX}(|x\rangle \otimes |y\rangle) = |x\rangle \otimes (|x\rangle \oplus |y\rangle)$$

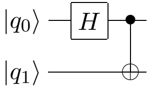
$$\mathrm{CX}\big(|0\rangle \otimes |\phi\rangle\big) = |0\rangle \otimes |\phi\rangle \qquad \mathrm{CX}\big(|1\rangle \otimes |\phi\rangle\big) = |1\rangle \otimes X|\phi\rangle$$

$$\mathrm{CX} = \mathrm{diag}(I, X) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Reversible operation ! $\mathrm{CX}^2 = I$

Lab 2

1. Consider the following quantum circuit:



   a) Starting from initial state $|00\rangle$, write the state of the qubits at each step.

   b) Confirm your computation by simulating and executing it on IBM Q.

2. What is the effect of this circuit?